# Phase synchronization of chaotic attractors with prescribed periodic signals

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Given a chaotic attractor in a dynamical system with dense periodic windows (i.e., structurally unstable), is it possible to find a periodic driver that will phase synchronize the chaotic attractor? We conjecture that the answer is typically yes, and we give an example for a funneling chaotic attractor in the Roessler system.

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## I. INTRODUCTION

Phase synchronization of low dimensional dynamical systems has been a field of active research recently. Numerous applications have been found, including plasma, laser, fluid, and biological experiments. Theoretical [1–5] and experimental [6] studies have been devoted to the synchronization of a chaotic attractor with the phase of an externally coupled harmonic signal. Phase synchronization of chaos in the presence of two harmonic signals has also been studied [7]. Phase synchronization of two coupled chaotic oscillators has received much attention as well, both in theory [8] and experiments [9,10].

In order to define phase synchronism, assume that we are given two signals $a$ and $b$, such that phases $\theta_a(t)$ and $\theta_b(t)$ can be defined for the two signals. The phases $\theta_{a,b}(t)$ are assumed to be continuous in time; i.e., they are not taken modulo $2\pi$. If for two times $t_2>t_1$, we have $\theta_{a,b}(t_2)-\theta_{a,b}(t_1)=2N\pi$, then we say that the phase $\theta_{a,b}$ has executed $N$ counter-clockwise rotations between time $t_1$ and $t_2$. In terms of the phase difference of the two time series $a$ and $b$, $\Delta\theta(t)=\theta_a(t)-\theta_b(t)$, there is phase synchronism between the signals $a$ and $b$ if

$$-C\leqslant\Delta\theta(t)-\theta_*\leqslant C$$

for some constants $C$ and $\theta_*$ (typically $C\sim\pi$) and all time $t$. Thus, $\Delta\theta$ does not increase or decrease without bound. This condition is also known in the literature as *strong phase synchronism* or *phase locking*.

If signal $a$, for example, is measured from a chaotic system, then defining a *good* phase $\theta_a(t)$ may be a challenging task. It is possible to define a phase for an attractor which in some projection appears to be shaped like an annulus. This type of attractor is called *phase coherent* [2]. The projection of an orbit on this attractor continually circles around a center of rotation in the hole of the annulus. A possible phase of an orbit on a phase coherent attractor can be defined using the polar angle about the center of rotation. Several examples of phase coherent attractors are known [2–5,8]. However, most attractors do not satisfy the requirements of phase coherence, and examples of attractors with ill-defined phase are also known [2,4]. These attractors are called *phase incoherent* attractors. In the Roessler system, *funneling* attractors [2,4] are phase incoherent.

In this work we discuss the entrainment of a typical attractor of a structurally unstable dynamical system which may not have a suitable phase. In the case of periodic driving, phase locking can be defined in a phase-free manner. We choose a point $P$ of the periodic driver, and, every time the periodic driver passes through $P$, we measure the chaotic attractor. If all these measurements are confined to a region of the chaotic attractor, we say that the periodic orbit locks the chaotic attractor (examples will be given subsequently). This definition of phase synchronization with a periodic signal does not depend on the particular choice of the point $P$.

In this paper we investigate the following question. Consider a dynamical system with a parameter $a$,

$$d\mathbf{x}/dt=\mathbf{R}(a;\mathbf{x}), \tag{1}$$

having dense periodic windows in the bifurcation diagram versus $a$. Given a value for the parameter $a$, where the system has a chaotic attractor, is it possible to find a periodic driver that will phase lock the chaotic attractor? We conjecture that the answer is typically yes for the dynamical systems having dense periodic windows. In this case, it is possible to find a periodic orbit with parameters in the vicinity of those of the chaotic attractor that will phase lock the chaotic attractor.

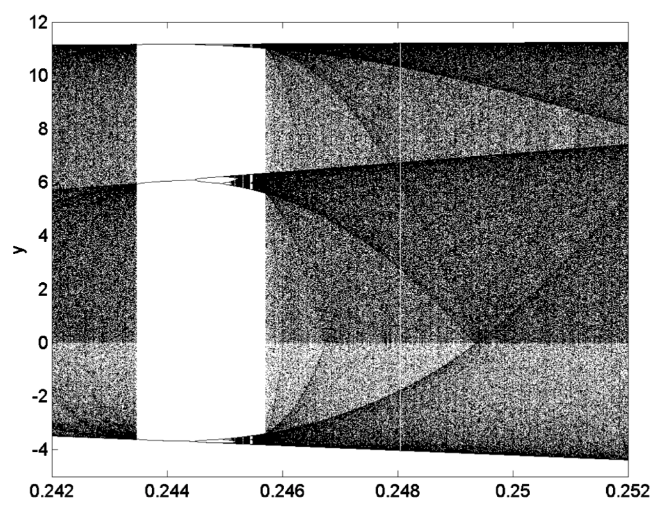


FIG. 1. Bifurcation diagram of the Roessler system described by Eq. 2. Our choice of chaotic attractor has $a=0.25$, and our choice of driver has $a=0.244$.

 

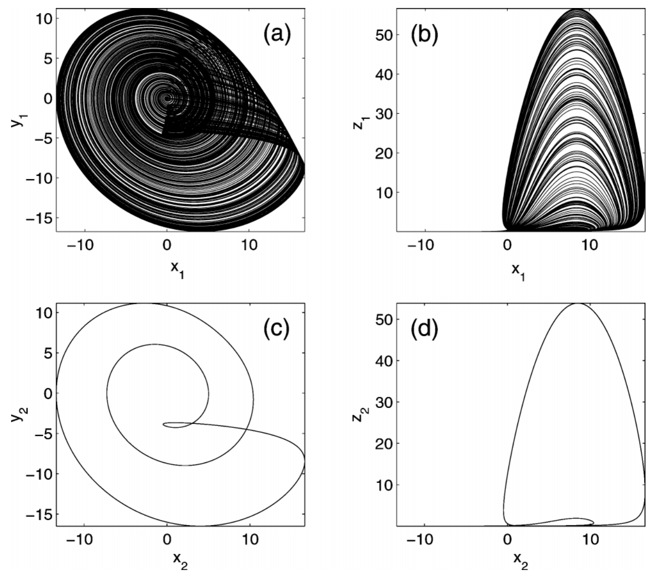


FIG. 2. (a) *x-y* projection of the chaotic Roessler system [see Eq. (2)] with a=0.25, and (b) the corresponding *x-z* projection. (c) and (d) show, respectively, the *x-y* and the *x-z* projections of our choice of periodic driver which is the Roessler system at $a=0.244$.

## II. MODEL

We numerically investigate the Roessler system for a set of parameters where a funneling attractor is observed [4]. Previous studies [4] point out that this attractor has an ill-defined phase, and that phase locking with a harmonic signal is not manifested. However, as we show in this study, driving the funneling attractor with an appropriate choice of periodic driver yields phase locking. Our model system is the Roessler attractor given by $\mathbf{x}^T(t)=(x(t),y(t),z(t))$, $\mathbf{R}^T(a;\mathbf{x})=(R_x(a;\mathbf{x}),R_y(a;\mathbf{x}),R_z(a;\mathbf{x}))$, and

$$R_x(a;\mathbf{x})=-y-z,$$

$$R_y(a;\mathbf{x})=x+ay, \tag{2}$$

$$R_z(a;\mathbf{x})=0.4+z(x-8.5).$$

Figure 1 shows a bifurcation diagram of our model system versus the parameter $a$. The bifurcation diagram is generated by plotting local maxima of $y(t)$ for orbits computed at different values of $a$. The funneling chaotic attractor subject to our investigation has $a=0.25$. Based on the bifurcation diagram in Fig. 1, we choose as periodic driving for our funneling attractor the Roessler system at $a=0.244$. Thus, we consider the following coupled dynamical systems

$$d\mathbf{x}_1/dt=\mathbf{R}(0.25;\mathbf{x}_1)+\epsilon(\mathbf{x}_1-\mathbf{x}_2), \tag{3}$$

$$d\mathbf{x}_2/dt=\mathbf{R}(0.244;\mathbf{x}_2), \tag{4}$$

where $\mathbf{x}_{1,2}^T=(x_{1,2}(t),y_{1,2}(t),z_{1,2}(t))$, $\epsilon(\mathbf{x}_1-\mathbf{x}_2)$ is a coupling term, and $\epsilon$ is a parameter describing the coupling strength. Many choices of coupling terms between the chaotic oscillator and the periodic driver are possible; we believe that our results apply to a large class of coupling terms. Figures 2(a,b) show *x-y* and *x-z* projections of the undriven funnel Roessler attractor, and Figs. 2(c,d) show *x-y* and *x-z* projections of the chosen driver.

## III. RESULTS

We investigate phase synchronization versus the coupling strength $\epsilon$ by sampling the chaotic attractor every time the periodic driver passes through its maximum value of $y_2(t)$. Figures 3(a1,b1,c1) show the *x-y* projection of the samplings of the chaotic attractor at the period of the driving (black dots), with the *x-y* projection of the chaotic attractor as the

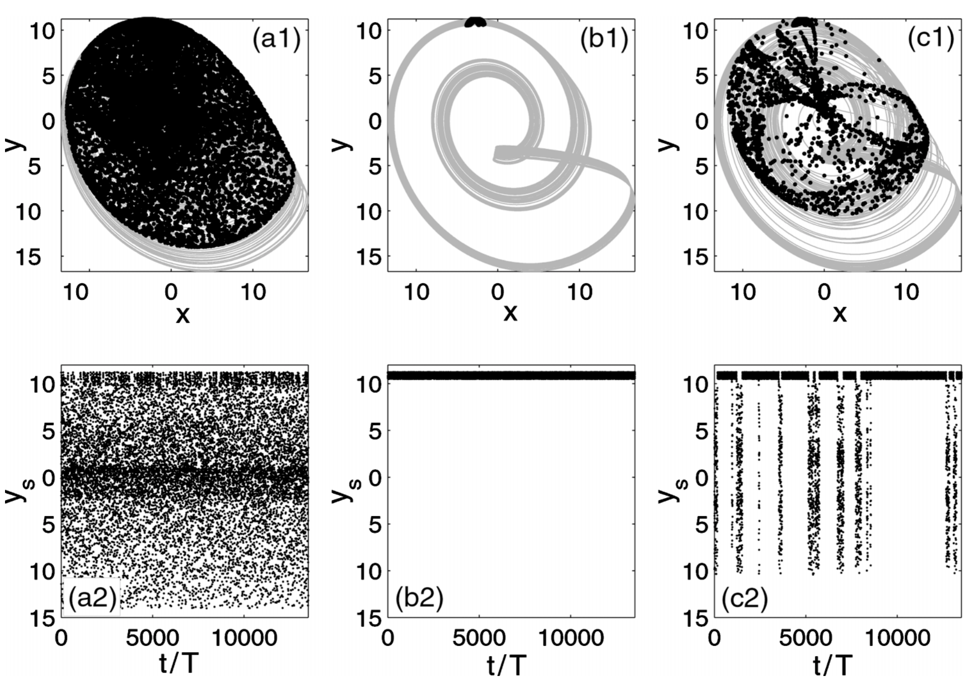


FIG. 3. *x-y* projection of the sampling at the period of the driver (black dots) of the entrained chaotic Roessler attractor (gray background) with (a1) $\epsilon=0.02$, (b1) $\epsilon=0.05$, and (c1) $\epsilon=0.0498$. Figures (a2), (b2), and (c2) represent the corresponding graphs of the *y* coordinate of the sampling map, $y_s$, vs time $t/T$, where $T$ denotes the period of the driving signal.

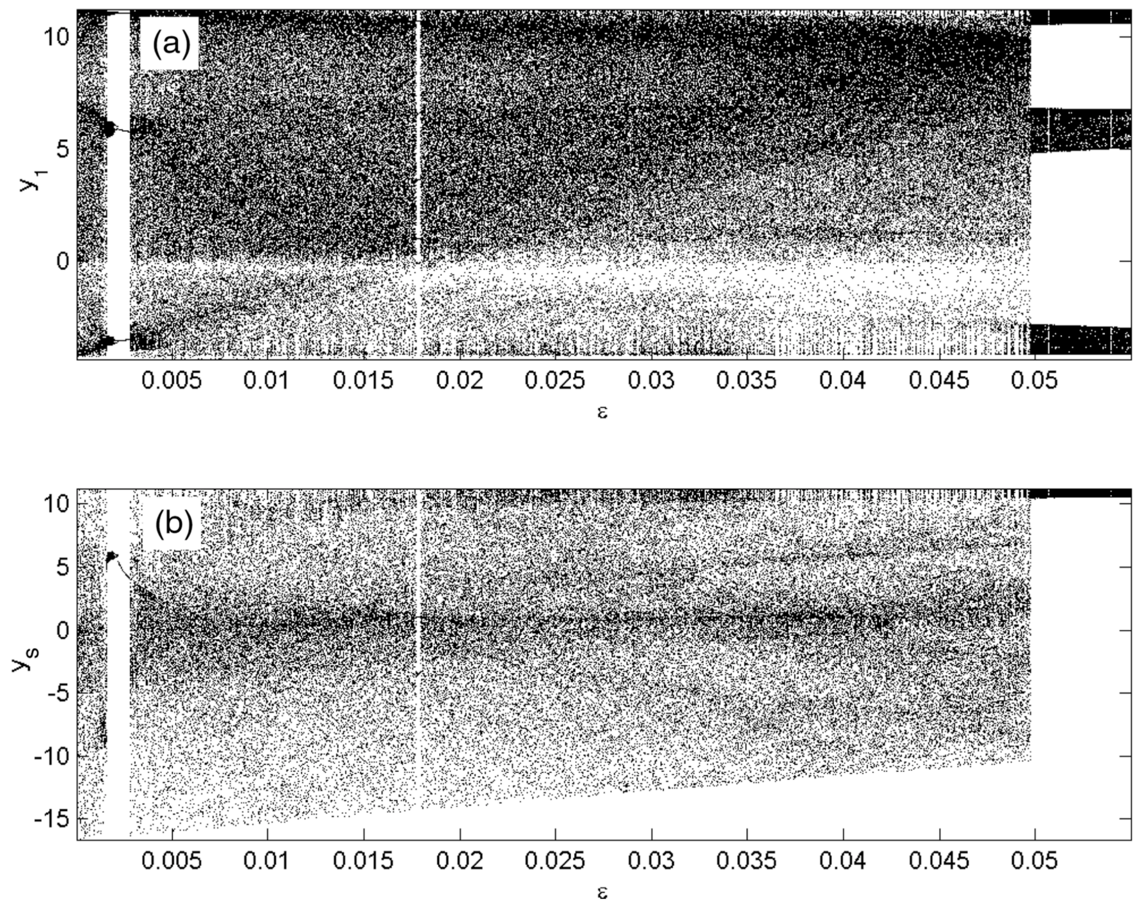


FIG. 4. Bifurcation diagram vs the coupling strength $\epsilon$ (a) of the entrained chaotic Roessler attractor, and (b) of the sampling map at the period of the driver.

gray background, for three different values of the coupling strength $\epsilon$. In Fig. 3(a1) we present the case of $\epsilon = 0.02$, where the sampling is spread over the whole attractor, and phase synchronization is not manifested. Figure 3(a2) shows the $y$ coordinate of the iterates of the sampling map, $y_s$ [i.e., the $y$ coordinates of the black dots in Fig. 3(a1)], versus time $t/T$, where $T$ denotes the period of the driver. The sampling for $\epsilon = 0.02$ remains spread throughout the attractor as time progresses. In Fig. 3(b1), corresponding to $\epsilon = 0.05$, phase synchronization for the perturbed attractor is clearly discernable. The sampling of the chaotic attractor at the period of the driver is localized in a small region. In this latter case, one may use for example the polar angle of the $x$-$y$ (or the $x$-$z$) projection to define phases for the periodic driver and the entrained chaotic attractor, and these phases would reveal locking of the two systems. Other definitions of phases [4,10] would also apply. Plotting the iterates of the sampling map versus time in Fig. 3(b2) we see that the sampling stays in a localized region of the chaotic attractor for all time.

As the perturbed attractor becomes phase synchronized with increasing coupling strength $\epsilon$, it also undergoes an unstable-unstable crisis that changes its shape, resembling the periodic driver [see Figs. 3(b1) and 2(c)]. How severely the crisis reshapes the chaotic attractor depends on the choice of the driver. Theoretically, for dynamical systems with dense periodic windows, one can find a periodic driver of high period which is, in the parameter space, arbitrarily close to the chaotic attractor. Such drivers would not induce very dramatic changes in the apparent shape of the chaotic attractor. The unstable-unstable crisis is known to play a crucial role in the phase synchronization of phase coherent attractors [3]. Here the unstable-unstable crisis is demonstrated in Figs. 3(c1,2) which correspond to $\epsilon = 0.0498$, just before the crisis. The sampling of the chaotic attractor is localized only for intervals of time [see Fig. 3(c2)] separated by brief excursions over large regions of the chaotic attractor. If the unstable-unstable crisis takes place at $\epsilon = \epsilon_c$, then the average duration of the synchronization intervals just before the crisis scales like $\langle \tau \rangle \sim (\epsilon - \epsilon_c)^{\gamma}$ [11]. The critical exponent $\gamma$ is negative, and its value depends on particular features of the attractor in unstable-unstable crisis.

Figure 4(a) shows the bifurcation diagram of the driven Roessler attractor [see Eq. 3] versus the coupling parameter $\epsilon$. As in Fig. 1, the bifurcation diagram is generated by plotting local maxima of $y_1(t)$ for orbits with different values of $\epsilon$. Figure 4(b) shows the bifurcation diagram of the map obtained by sampling the chaotic attractor at the period of the driver. (When the driver passes through the maximum value of $y_2(t)$, we take a measurement of the chaotic attractor.) At $\epsilon \approx 0.05$, a crisis reshapes the chaotic attractor to three-band chaos, and simultaneously, phase synchronization occurs (see Fig. 4). The sampling of the chaotic attractor past the crisis is localized in the upper band; this corresponds to the situation described in Fig. 3(b1). A similar scenario applies at $\epsilon \approx 0.002$, where again the chaotic attractor undergoes crisis, and simultaneously, phase synchronization. In this latter case, however, the sampling of the chaotic attractor at the period of the driver (as the driver passes through its maximum value of $y_2$) is localized in the center band. Thus, in this phase synchronized state, the difference between the phase of the driver and the phase of the entrained chaotic attractor is not small, but nevertheless bounded. In fact, since our model system has dense periodic windows, we believe that one can find such windows (where crisis and phase synchronization occur simultaneously) arbitrarily close to $\epsilon = 0$. In practice, noise may wash out some of the small periodic window structure, such that only large windows are discernable. Figure 4 plays the role of the "synchronization tongue" plot for our coupled dynamical systems, and depends on the choice of the coupling term between the chaotic attractor and the periodic driver.

In conclusion, we have shown that phase synchronization of chaos is a general phenomenon, not restricted to the class of the phase coherent chaotic attractors. Phase synchronization becomes manifest for chaotic attractors of structurally unstable dynamical systems driven by selected periodic drivers. In practice, finding an appropriate driver signal may not be a very difficult problem. As we demonstrate numerically, the same dynamical system for a different set of parameters may have a periodic orbit that successfully phase synchronizes the chaotic attractor.

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